বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. General Examinations 2021

(Under CBCS Pattern)

Semester - V

Subject : MATHEMATICS

Paper : DSE 1A/2A/3A - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[COMPLEX ANALYSIS]

Group-A

Answer any ***four*** questions : 12×4=48

1. (a) State and Proof Cauchy-Riemann Equations in Complex Analysis. 6

   (b) Let $f(z)=(x^3+2)+i(1-y)^2$. Find all the points in the complex plane where $f(z)$ is differentable and compute $f'(z)$ of those points. Is $f(z)$ analystic of any point in the complex plane? Justify. 6

2. (a) Prove that $u=e^{-x}(x\,sin\,y-y\,cos\,y)$ is harmonic. 6

(b) Find the order of the pole of $z = \frac{\pi}{4}$ of $f(z) = \frac{1}{\cos z - \sin z}$ 6

3. (a) Show that the unction $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the point. 4

(b) Find the residue of $f(z) = \frac{7z-2}{(z+1)^2(z-2)}$ at its poles. 8

4. (a) Show that the function $f(z) = \frac{z}{e^z - 1}$ has a removal singularity at the origin. 6

(b) Prove the Cauchy-Goursat theorem or any simple closed curve. 6

5. (a) State and Proof Liouville's theorem. 6

(b) Prove that the function $f(z)$ given by

$f(z) = \begin{cases} Im\, z / |z| \ , & if \ \ z \neq 0 \\ 0 \ , & if \ \ z = 0 \end{cases}$ is not continuary at z = 0 6

6. (a) State and Proof Laurent's theorem. 6

(b) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series for | z | < 1. 6

7. (a) Evaluate $\frac{1}{2\pi i} \oint \frac{e^{zt}}{z^2(z^2+2z+2)}\, dz$ around the circle C with equation | z | = 3. 6

(b) Evaluate $\oint\limits_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\, dz$ by means of the Cauchy residue theorem. 6

8. (a) State and Prove Taylor's series in complex field. 10

(b) Let $f(z) = \sqrt{z}, z \in c$. Test whether $f(z)$ is analytic or not at origin. 2

Group-B

Answer any ***six*** questions : 2×6=12

1. Using Cauchy's integral formula, evaluate the integral $\int_{|z|=1} \frac{cos(2\pi z)\,dz}{(2z-1)(z-2)}$, where $|z| = 1$ is positively oriented circle.

2. State and proof fundamental theorem of integral calculus in complex plane.

3. Evaluate $\int_C log z \; dz$, where C is unit circle $|z| = 1$.

4. Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series about z = 0.

5. Find residue of $\phi(z) = cot\, z$ at the point $z_n = n\pi$ for $n = 1, 2,.....$

6. Evaluate by method of calculus of residues : $\int_C \frac{dz}{(z^2+1)(z-4)}$, where C is a circle $|z| = 3$.

7. Find the radii of convergence of power series $\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n$.

8. Find the domain of convergence of power series $\sum \left(1-\frac{1}{n}\right)^{n^2} z^n$.

9. If $f(z) = z^2$ then prove that $lim_{z \to z_0} f(z) = z_0^2$.

10. Prove that $lim_{z \to 0} \frac{\overline{z}}{z}$ does not exist?

OR

[MATRICES]

Group-A

Answer any ***four*** questions : 12×4=48

1. (a) Examine whether the set of vectors {(1, 2, 2), (2, 1, 2), (2, 2, 1)} is linearly dependent or independent in R^3.

   (b) If A and P be both $n \times n$ matrices and p be non-singular, then prove that A and $P^{-1}AP$ have the same eigen values. 6+6

2. (a) Use elementary row operations on the following matrix to obtain its inverse :

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 6 & 4 & 1 \end{pmatrix}$$

   (b) Prove that there exists a basis for every finitely generated vector space. 6+6

3. (a) Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   (b) Find a matrix P such $P^{-1}\,AP$ is a diagonal matrix, where $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

4. (a) If $V = R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$ and $F = R$, then show that R^2 is a vector space over R with pointwise addition and scalar multiplication defined by $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$.

   (b) If A and B are square matrices of order n, then prove that $\rho(AB) \geq e(A) + \rho(B) - n$, [where $\rho(A)$ means row rank of A]

5. (a) Find the inverse of the matrix $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

   Hence obtain the solution of the equation AX = B, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

(b) Prove that eigen vectors corresponding to distinct eigen values are linearly independent. 6+6

6. (a) If $W = \{(x, y, z) \in R^3 : x - 4y + 3z = 0\}$, then show that w is a subspace of R^3.

(b) Find all eigen values and corresponding eigen spaces or the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Diagonalise A, if possible. 6+6

7. (a) Find the normal form under congruence and obtain the rank and signature of the symmetric matrix $\begin{bmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{bmatrix}$.

(b) If *v* be a real vector space with $\{\alpha, \beta, \gamma\}$ as a $\{\alpha + \beta + \gamma, \beta + \gamma, \gamma\}$ is also a basic of *v*. 6+6

8. (a) Solve the system of equations $x + 2y + z = 1$, $3x + y + 2z = 3$, $x + 7y + 2z = 1$ in integers.

(b) Prove that an $n \times n$ matrix A over a field F is diagonalisable if and only if there exist n eigen vectors of A which are linearly independent. 6+6

Group-B

Answer any ***six*** questions : 2×6=12

9. (a) Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$.

(b) If S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$, show that s is a subspace of $\mathbb{R}^3$.

(c) In $\mathbb{R}^3$, $\alpha = (4, 3, 5), \rho = (0, 1, 3), \gamma = (2, 1, 1)$. Examine whether α is a linear combination of β and γ ? Justify.

(d) Using row transformation, find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(e) Find the eigen values of the matrix $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$.

(f) Find the value of m so that the vector (m, 3, 1) is a linear combination of the vectors (3, 2, 1) and (2, 1, 0).

(g) Using matrix method, solve the equations $2x + 3y = 5,\ 4x - 7y + 3 = 0$.

(h) Show that the rank of the transpose of a matrix is the same as that of the original matrix.

(i) Prove that the eigen values of a diagonal matrix are its diagonal elements.

(j) In a vector space V over a field F, prove that $c\alpha = \theta$ implies either $c = 0$ or $\alpha = \theta$.

OR

[LINEAR ALGEBRA]

Group-A

1. Answer any ***four*** questions : 12×4=48

(a) (i) If U, W be two subspaces of a vector space V over a field F, then prove that the union $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.

(ii) Show that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is linearly independent in R (set of real numbers) over Q (set of rational numbers). 6+6

(b) (i) Find dimension of $S \cap T$, where S and T are subspaces of the vector space R^4 given by $S = \{(x, y, z, w) \in R^4 : 2x + y - z + w = 0\}$

$$T = \{(x, y, z, w) \in R^4 : x + y + z + w = 0\}$$

(ii) Let V and W be vector spaces over a ield . Let $T : R^3 \to R^3$ be a transormation deined by $T(x,y,z) = (x+y+z,\ 2x+y+2z,\ x+2y+z)$. Prove that T is linear. Find the kernel of T and dimension of Ker (T). 6+6

(c) (i) Find a basis or the vector space R^3, that contains the vectors (1,2,1), (3,6,2).

(ii) Find a basis and the dimension of the subspace S of the vector space $R_{2\times2}$, where

(a) S is the set of all 2×2 real diagonal matrices;

(b) S is the set of all 2×2 real symmetric matrices.

(d) (i) For what values of $k \in R$ does the set $S = \{(k,1,1), (1,k,1), (1,1,k)\}$ form a basis of R^3?

(ii) Let $T : R^3 \to R^3$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z, 2x + 3y + z, 3x + y + 2z).$$

Find the matrix of T relative to the ordered bases (–1, 1, 1), (1, –1, 1), (1, 1, –1) of R^3. 6+6

(e) (i) Show that the set S={(1,2,3,0), (2,3,0,1), (3,0,1,2)} is linearly dependent in R^4.

(ii) A linear mapping $T : R^3 \to R^3$ maps the vectors (0,1,1), (1,0,1), (1,1,0) to the vectors (1,1,–1), (1,–1,1), (1,0,0) respectively. Show that T is not an isomorphism. 6+6

(f) (i) Find the dimension of the subspaces S of R^4 given by

$S = \{(x, y, z, w) \in R^4 : x + 2y - z = 0,\ 2x + y + w = 0\}$

(ii) Lev V and W be two finite dimensional vector spaces over a field F. Prove that they are isomorphic if and only i dim V = dim W. 6+6

(g) (i) Prove that the intersection of two subspaces of a vector space V over a field F is a subspace of V. Is the union of two subspaces of V a subspace of V? Justify.

(ii) Determine the subspace of R^3 spanned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if $\gamma = (2,1,3), \delta = (-1,3,6)$ are in the subspace.

(4+3)+5

(h) (i) Prove that the subset $D[a,b]$ of all real valued diferentiable functions on $[a,b]$ is a subspace of $C[a,b]$.

(ii) Determine the linear transformation $T : R^3 \to R^2$ which maps the basis vectors (1,0,0), (0,1,0), (0,0,1) of R^3 to the vectors (1,1), (2, 3), (3, 2) of R^3. Find T(1,1,0) and T(6, 0, –1). 6+6

Group-B

2. Answer any ***six*** questions : 2×6=12

(a) Define isomorphism of a linear transformation.

(b) Define linear dependence ad linear independence of a set of vectors.

(c) Define inverse of a linear transformation.

(d) Examine if the set S is a subspace of the vector space $R_{2\times2}$, where S is the set of all 2×2 real diagonal matrices.

(e) Define kernel and image of a linear mapping.

(f) Let $S=\{(x,y,z)\in R^3 : x=y=z\}$, Prove that S is a subspace of R^3.

(g) Prove that the set of vectors

$\{(1, 1, 0),(1,3,2),(4,9,5)\}$ is linearly dependent in R^3.

(h) Let T : $R^3 \rightarrow R^2$ be a linear trasformation defined by $T(x, y,z)=(x, y, 0)$. Find the kernel and image of T.

(i) Consider the linear operator $T:R^2 \rightarrow R^2$ defined by $T(x,y)=(2x-7y, 4x+3y)$ and $S=\{(1,3),(2,5)\}$ be a basis on R^2. Find the matrix representation of T relative to S.

(j) In a vector space V over a field F, prove that $0.\alpha=0$ for all $\alpha\in V$.

OR

[VECTOR CALCULUS & ANALYTICAL GEOMETRY]

Group-A

Answer any ***four*** questions : 12×4=48

1. (a) Prove that a vector function $\vec{F}(t)$ has constant magnitude if and only if $\vec{F}(t)\cdot\frac{d\vec{F}(t)}{dt}=0$.

   (b) Prove that $\vec{r}.\frac{d\vec{r}}{dt}=|\vec{r}|\frac{d}{dt}|\vec{r}|$.

   (c) If $\vec{F}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$ and $\vec{G}=(2t-3)\hat{i}+\hat{j}-t\hat{k}$, then find $\frac{d}{dt}\left(\vec{F}\times\frac{d\vec{G}}{dt}\right)$ at t = 1.

   6+3+3=12

2. (a) If $\varphi(x,y,z)=xyz$ and $\vec{F}(x,y,z)=xz^2\hat{i}-x^2y\hat{j}+yz\hat{k}$, find $\frac{\partial}{\partial x}(\varphi\vec{F})$ and $\frac{\partial^2(\varphi)}{\partial x\,\partial z}$ at (0, 0, 1)

   (b) If $\vec{F}(x,y,z)x^2y\hat{i}+y\hat{j}+xyz\hat{k}$ and $x=z^2,\ y=3z$, then find $\frac{d\vec{F}}{dz}$.

   (c) A particle moves along a curve $x=e^{-t},\quad y=2\cos 3t,\ z=2\sin 3t$, where t is the time. Determine its velocity and acceleration at t = π.

   4+4+4=12

3. (a) If $\vec{a}$ is a constant vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then prove that curl $(\vec{r}\times\vec{a})=-2\vec{a}$.

   (b) Find the value of $\nabla^2(r^n\vec{r})$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ of $r=|\vec{r}|$.

   (c) If $\vec{\nabla}\varphi=(y^2-2xyz^3)\hat{i}+(3+2xy-x^2z^3)\hat{j}+(6z^3-3x^2yz^2)\hat{k}$, then find φ.

4. (a) P be an only point on an ellipse whose two fociii are s and s'. If TPT' is tangent at P then prove that $\underline{|TPS'}=\underline{|T'PS}$.

(b) Reduce the equation $7x^2 - 6xy - y^2 + 4x - 4y - 2 = 0$ to its cononical form and find the nature of the conic.

5. (a) Find the values of C for which the plane $x + y + z = e$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

(b) Find the equation of the cylinder whose guiding curve is the circle through the points (1,0,0), (0,1,0) and (0,0,1)

(c) If the equation $ax^2 + by^2 + cz^2 - 4ax + 3by - 6cz + 5 = 0$ represents a sphere for suitable values of *a, b, c*; find its centre.

6. (a) Find the equation of the right circular cylinder which passes through (3, –1, 1) and has the axis $\frac{x-1}{2} = \frac{y+3}{-1} = z - 2$.

(b) Find the equation of the sphere having its centre on line $5y + 2z = 0 = 2x - 3y$ and passing through the points (0, –2, –4) and (2, –1, 1).

7. (a) Sketch the graph of $9x^2 + 16y^2 = 144$.

(b) Correct or justify the statement :

$x^2 + xy - 2y^2 + 3y - 1 = 0$ represents an ellipse.

(c) Find the centre and the radius of the circle given by $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$, $x + 2y + 2z = 15$

6+4+2=12

8. (a) Prove by vector method that if two medians of a triangle be equal then the triangle is isosceles.

(b) If $\vec{r} = (x\cos y)\vec{i} + (x\sin y)\hat{j} + (2e^{my})\hat{k}$ find $\left[\frac{\partial \vec{r}}{dx}\ \frac{\partial \vec{r}}{\partial y}\ \frac{\partial^2 \vec{r}}{\partial x\,\partial y}\right]$

(c) Find curl grad ϕ, for any scalar function ϕ.

Group-B

Answer any ***six*** questions : 2×6=12

9. (a) Detemine the nature of the locus given by $x^2+6xy+9y^2-5x-15y+6=0$.

(b) Sketch the graph of $y=\sqrt{x}, x\geq 0$.

(c) Find the equation of the sphere which has (3, 4, –1) and (4, 2, 3) as the end points of a diameter.

(d) If $\frac{d^2\vec{r}}{dt^2}=\vec{r}$ then show that $\vec{r}\times\frac{d\vec{r}}{dt}$ is a constant vector.

(e) Are these three vectors $4\hat{i}+2\hat{j}+\hat{k}, 2\hat{i}-\hat{j}+3\hat{k}$ and $8\hat{i}+7\hat{k}$ co-planar? Justiy your answer.

(f) If $\vec{F}(t)=(e^{-t})\hat{i}+log(1+t^2)\hat{j}-(tan\,t)\hat{k}$, find $\left|\frac{d^2\vec{F}}{dt^2}\right|$

(g) If $\vec{F}=(2xy-x^2)\hat{i}+(e^{2xy}-y\,cos\,x)\hat{j}+(x^2\,sin\,y)\hat{k}$ find $\frac{\partial^2\vec{F}}{\partial x^2}$ and $\frac{\partial^2\vec{F}}{\partial x\,\partial y}$.

(h) Find the equation of the right circular cylinder whose radius is 1 and axis is the x-axis.

(i) Find the equation of the sphere passing through the points (0,0,0), (2,0,0), (0,3,0) and (0,0,4).

(j) If $\vec{A}=grad(x^3+y^3+z^3-3xyz)$, then find div $\vec{A}$ and curl $\vec{A}$.